SYLLABUS FOR MATH 8120: THE ARITHMETIC OF ELLIPTIC CURVES (SPRING 2025)

TYLER GENAO

Contents

1. Instructor Details

1. **Instructor:** Tyler Genao
2. **Office:** MW 746 (Mathematics Tower)
3. **Email:** genao.5@osu.edu
4. **Classroom and times:**
   - Enarson Classroom Building, Room 312
   - Monday/Wednesday/Friday, 1:50 - 2:45 PM
   - Homework presentations: Friday, 5 - 6 PM in MW 154.
5. **Office Hours:** by request.

2. Course Description

2.1. **Introduction.** This is a graduate level number theory topics course on **the arithmetic of elliptic curves.** This course will be an introduction to the topic; we will primarily follow Joseph Silverman's *The Arithmetic of Elliptic Curves,* second edition.

Date: Last updated April 22, 2025.

What is an elliptic curve? The most common definition of an *elliptic curve over a field k* is a smooth projective curve of genus one, equipped with a k-rational point. Elliptic curves are often given by an equation of the form $y^2 = x^3 + Ax + B$ where $A, B \in k$. As it turns out, the set $E(k)$ of k-rational points on E is a group, and when E is a plane cubic, this group operation can be described by a succinct chord and tangent method.

Studying rational points on elliptic curves is a special case of studying rational solutions to Diophantine equations. However, since an elliptic curve has a group law, it also admits unique algebraic and arithmetic properties that can be used to study these points. Elliptic curves have a central place in modern number theory: for example, they are an essential part of the proof of Fermat's Last Theorem.

2.2. **Structure of class content.** The main goal of this class is to get a better foundational understanding of the group $E(k)$ of k-rational points on an elliptic curve, particularly when k is a number field. In this case, we have by the Mordell-Weil Theorem that $E(k)$ is a finitely generated abelian group. Thus, to better understand $E(k)$, one must understand both its torsion subgroup and rank; these are active areas of research. In this class, however, we will learn some foundational techniques that can be used to study $E(k)$, such as analyzing points over local fields, as well as its Tate-Shafarevich group.

The specific topics we cover are subject to change, but at the moment are the following. These are roughly divided into three "sections" of the course; I've included chapter numbers from the textbook when applicable.

1. **The geometry of elliptic curves:**
   - *Chapter zero, elliptic curves from a planar perspective:* defining elliptic curves in $\mathbb{A}^2$ and $\mathbb{P}^2$, as well as the group law via the chord and tangent method.
   - *Chapter 1, algebraic varieties:* affine and projective varieties, and their morphisms.
   - *Chapter 2, algebraic curves:* curves, their morphisms, divisors and differentials. The Riemann-Roch theorem.
   - *Chapter 3, the geometry of elliptic curves:* the basic theory of elliptic curves. This includes Weierstrass equations, the geometric and Picard group law, isogenies, the invariant differential, the dual isogeny, the Tate module, the Weil pairing, and elliptic curve endomorphisms and automorphisms.
   - *Chapter 5, elliptic curves over finite fields:* we will review the Hasse-Weil bound.
2. **Elliptic curves over local fields:**
   - *Chapter 7, elliptic curves over local fields:* elliptic curves over local fields and reduction modulo π, points of finite order, the action of inertia, good and bad reduction, and the Néron-Ogg-Shafarevich criterion.
3. **The Mordell-Weil group of an elliptic curve over a number field:**
   - *Chapter 8, elliptic curves over global fields:* the (weak) Mordell-Weil theorem, the Kummer pairing, Galois cohomology, descent, and heights.

- *Chapter 10, computing the Mordell-Weil group:* complete 2-descent with an example, and a brief description of the Selmer and Tate-Shafarevich groups.

2.3. **Textbook.** We will primarily follow Joseph Silverman's *The Arithmetic of Elliptic Curves,* second edition. You are not required to buy this book, since all class notes will be typed up and provided before each lecture.

2.4. **Prerequisite.** Algebraic number theory (MATH 7121.01 or 7121.02) is required, or its equivalent. An introductory course in algebraic geometry (such as MATH 7141.01 or 7141.02) can be very helpful for the first half of the class; however, we will review the algebraic geometry we need from Chapters 1 and 2. Note that it will be possible to "black box" the main algebro-geometric results from Chapter 2, and still learn a lot about elliptic curves from this course.

3. Class Calendar

The following calendar is a rough outline of the topics we will cover this semester – this is subject to change! The numbering will follow the sections from the textbook, sans including Chapter 0. Note the **bold** dates, which imply a holiday or canceled lecture that week.

Week	Sections	Reminders
January **8** – 10	Introduction, Chapter 0	Surveys due Friday
January 13 – 17	Chapter 0	
January **22** – 24	1.1, 1.2	**MLK day (1/20)**
January 27 – 31	1.2, 1.3, 2.1, 2.2	
February 3 – 7	2.2, 2.3, 2.4	
February 10 – 16	2.5, 3.1, 3.2, 3.3	
February 17 – 21	3.3, 3.4, 3.5	
February 24 – 28	3.5, 3.6, 3.7, 3.8	
March 3 – 7	3.8, 3.9, 3.10, 5.1	
March 10 – 14		**Spring break**
March 17 – 21	7.1, 7.2, 7.3 (& 3.2)	
March 24 – 28	7.4, 7.5, 7.6, 7.7, 8.1	Final project choice due Friday
March 31 – April 4	8.1, 8.2 (& Appendix B)	
April 7 – 11	8.3, 8.5, 8.6	
April 14 – 18	8.9, 10.1	
April 21	10.1, 10.4 (brief)	Last day of class
April 29		Final presentations

4. Assessments

Your grade will be determined by the following 4 components:

1. Your completion of the initial class survey.
2. Your attendance of the lectures and presentations.
3. The quality of your homework solution presentations.

4. The quality of your final project presentation.

4.1. **Initial class survey.** In the first week of class, there will be an "Initial Survey" on Carmen due by Friday, January 10. The main goal of this survey is to help me get to know you better.

4.2. **Attendance.** Your attendance of lecture and presentations is expected, and will be part of your final grade. If you can't attend a particular lecture or presentation session, you should let me know as soon as you can.

4.3. **Homework presentations.** In this class, there will be no written homework to submit. Instead, each week we will have $3-4$ people volunteer to work on HW problems, to present the following week. These problems will be from a continuously updating Overleaf file, where I will add new problems as we progress through the course.

Each HW solution presented should be ~ 15 minutes in length. Each enrolled student can expect to present on $\sim 3-4$ problems in total the entire semester. For the HW presentations, we will meet during an hour separate from the lectures, which we will determine in the first week with a survey (for all enrolled students).

4.4. **Final project presentations.** Instead of having exams in this course, we will have project presentations at the end of the semester. This is separate from the HW presentations: these will be $25-30$ minutes in length, and will be on a topic that isn't fully covered in class.

Around the middle of the semester, I will release a list of topics you can choose from for your project. You will have approximately a month to work on it. If you would like, you can work in pairs for your presentation. We will have project presentations on the final exam day (Tuesday, April 29 from 4 - 5 : 45 PM), as well as on 1 or 2 additional days.

5. Additional Resources

5.1. **Disabilities.** The university strives to maintain a healthy and accessible environment to support student learning in and out of the classroom. If you anticipate or experience academic barriers based on your disability (including mental health, chronic, or temporary medical conditions), please let me know immediately so that we can privately discuss options. To establish reasonable accommodations, I may request that you register with Student Life Disability Services. After registration, make arrangements with me as soon as possible to discuss your accommodations so that they may be implemented in a timely fashion. SLDS contact information: slds@osu.edu; 614-292-3307; slds.osu.edu.

5.2. **Mental Health.** As a student you may experience a range of issues that can cause barriers to learning, such as strained relationships, increased anxiety, alcohol/drug problems, feeling down, difficulty concentrating and/or lack of motivation. These mental health concerns or stressful events may lead to diminished academic performance or reduce a student's ability to participate in daily activities. The Ohio State University offers services to assist you with addressing these and other concerns you may be experiencing.

If you or someone you know are suffering from any of the aforementioned conditions, you can learn more about the broad range of confidential mental health services available on campus via the Office of Student Life's Counseling and Consultation Service (CCS) by visiting ccs.osu.edu or calling 614-292-5766. CCS is located on the 4th floor of the Younkin Success Center and 10th floor of Lincoln Tower. You can reach an on-call counselor when CCS is closed at 614-292-5766 and 24-hour emergency help is also available through the 24/7 by dialing 988 to reach the Suicide and Crisis Lifeline.